

Black hole discharge in massive electrodynamics and black hole disappearance in massive gravity

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Abstract

We define and calculate the *discharge mode* for a Schwarzschild black hole in massive electrodynamics. For small photon mass, the discharge mode describes the decay of the electric field of a charged star collapsing into a black hole. We argue that a similar “discharge of mass” occurs in massive gravity and leads to a strange process of black hole disappearance.

1 Introduction and Estimates

In massive electrodynamics, a charged star collapsing into a black hole must lose its electric field – the black hole must discharge. For a small photon mass m , the rate of discharge can be estimated as follows. One treats the mass term $m^2\phi$, where $\phi \approx q/r$ is the electrostatic potential, as the density of the screening charge $\rho_s = -m^2\phi/(4\pi)$. One further assumes that the screening charge moves into the black hole at the speed of light. Then $\dot{q} \approx 4\pi r_g^2 \rho_s = -qm^2 r_g$, where r_g is the Schwarzschild radius. This gives an exponential discharge, $q \propto e^{-\gamma t}$, with the decay rate $\gamma \approx m^2 r_g$. In §2 we show that this estimate correctly describes the (intermediate asymptotic) decay of the electric field near the black hole in the limit of small photon mass.

Now consider a black hole in massive gravity. The gravitational field of the black hole is screened by a negative energy density $\rho_s \sim -m^2 M_P^2$, where m is the graviton mass and M_P is the Planck mass. We argue in §3 that this negative energy must be accreted onto the black hole. Assuming, like we did in massive electrodynamics, that the screening energy accretes onto the black hole at the speed of light, we get a decreasing black hole mass: $\dot{M} \sim -r_g^2 m^2 M_P^2$. Thus, the black hole loses mass and gradually disappears.

The black hole disappearance is a weird prediction of massive gravity, as it seems to make possible the following scenario: (i) there was a star in an asymptotically Minkowski space-time, (ii) the star collapses into a black hole, (iii) the black hole disappears leaving behind just the Minkowski space-time.

To be clear, our results are inconsequential for real astrophysical black holes. If we live in an asymptotically flat universe with a massive graviton, the graviton mass must be smaller than the Hubble constant and the disappearance time much longer than the Hubble time. In fact, the phenomenological continuity of $m \rightarrow 0$ limit is a quite satisfactory outcome. But

the very possibility of disappearance (from the asymptotically Minkowski, eternal universe) seems to be a badness of the massive gravity.

What makes the above scenario impossible in General Relativity is the conservation of the asymptotically defined ADM mass. In massive gravity the ADM mass vanishes – this is the characteristic property of the theory. And while an alternative globally conserved quantity can be defined in massive gravity (§3.5), global charges are not conserved in the presence of black holes.¹

2 Black hole discharge in massive electrodynamics

In this section, we first formulate the Einstein-Proca theory and argue that charged black holes must discharge (§2.1). We next study the decay of the electric field and define the discharge mode (§2.2). Finally, we find quasi-stationary time-dependent solutions in the limit of small photon mass and identify the discharge mode (§2.3). Our massive gravity calculation will closely follow the calculation of §2.3.

2.1 Einstein-Proca theory. Singularity of static charged black hole solutions

The field of charged stars and black holes in massive electrodynamics is governed by the Proca equation

$$\partial_\nu(\sqrt{-g}g^{\nu\alpha}g^{\mu\beta}F_{\alpha\beta}) + m^2\sqrt{-g}A^\mu = 4\pi\sqrt{-g}J^\mu, \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and J^μ is the electric current. The Einstein equation is

$$G_{\mu\nu} = \mathcal{T}_{\mu\nu} + T_{\mu\nu}^{(m)}, \quad (2)$$

where $\mathcal{T}_{\mu\nu}$ is the stress-energy tensor of matter and $T_{\mu\nu}^{(m)}$ is the stress-energy tensor of the Proca field

$$T_{\mu\nu}^{(m)} = T_{\mu\nu}^{(0)} + m^2(A_\mu A_\nu - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}A_\alpha A_\beta), \quad (3)$$

where $T_{\mu\nu}^{(0)}$ is the stress tensor of the Maxwell theory. For black holes and outside stars $J^\mu = 0$, $\mathcal{T}_\nu^\mu = 0$. Here, we only consider the spherically symmetric case where the metric is diagonal in (t, r, θ, φ) coordinates. The spherical symmetry also implies that the only non-vanishing component of $F_{\mu\nu}$ is F_{tr} .

¹After submitting the first version of this paper, we were informed of the work [1] on black holes in bimetric gravity. There, it has been argued that asymptotically flat static black holes are non-singular only when the two metrics coincide with each other, and hence, with the Schwarzschild solution of the Einstein gravity. Thinking of massive gravity as a bimetric theory in the limit where the Newton’s constant \tilde{G} of the second metric vanishes, the gravitational radius with respect to this metric is always zero, leading to the full disappearance of black holes in massive gravity. For any finite \tilde{G} , we expect the “mass discharge” of black holes in bimetric gravity to continue until the gravitational radii with respect to the two metrics coincide, at which point the time-dependent black hole solutions settle to the regular static solutions.

For static solutions, the only non-zero component of the Proca field is A_t . The A_r component vanishes, because the $\mu = r$ component of equation (1) reads

$$\partial_t(\sqrt{-g}g^{tt}g^{rr}F_{tr}) + m^2\sqrt{-g}g^{rr}A_r = 4\pi\sqrt{-g}J^r, \quad (4)$$

which forces $A_r = 0$ when $\partial_t = 0$ and $J^r = 0$. The $\mu = t$ component of (1) then becomes

$$\partial_r(\sqrt{-g}g^{rr}g^{tt}\partial_r A_t) + m^2\sqrt{-g}g^{tt}A_t = 4\pi\sqrt{-g}J^t, \quad (5)$$

and gives A_t . Solutions vanishing at spatial infinity have the expected Yukawa behavior at large r :

$$A_t \approx \frac{qe^{-mr}}{r} \quad \text{when} \quad r \gg r_g. \quad (6)$$

Defining the electric charge via the Gauss's law, this shows that localized sources are screened by the opposite charge density $-m^2 A^t/4\pi$, carried by the Proca field.

This is how one can calculate the electric field of a charged star. But static charged black holes do not exist, as they would be singular at horizon. To demonstrate this, consider the invariant product $g^{\mu\nu}A_\mu A_\nu$, which is an observable in Proca theory. The product diverges at horizon because it reduces to $g^{tt}A_t^2$ and $A_t \neq 0$ at horizon (if A_t were to vanish both at horizon and at infinity, eq.(5) would give $A_t(r) = 0$, corresponding to an uncharged black hole).

The horizon singularity of the static solution indicates that newly formed charged black holes would get rid of their electric field hair. The corresponding time-dependent solutions will be studied in the next section.

2.2 Non-singular time-dependent black holes and the discharge mode

For simplicity, we assume that the charge is small. Then we can use the Schwarzschild metric

$$ds^2 = (1 - \frac{r_g}{r})dt^2 - (1 - \frac{r_g}{r})^{-1}dr^2 - r^2d\Omega^2. \quad (7)$$

The vacuum Proca equation for spherically symmetrical field can be transformed to an equation for the physical electric field $E = \sqrt{-F^2/2}$ (the field is, of course, radial)

$$\partial_t^2 E - (1 - \frac{r_g}{r})\partial_r \left((1 - \frac{r_g}{r}) \frac{1}{r^2} \partial_r (r^2 E) \right) + m^2 (1 - \frac{r_g}{r}) E = 0, \quad (8)$$

which can be brought to another useful form:

$$(\partial_t^2 - \partial_\rho^2 + V)\Psi = 0, \quad (9)$$

where $\Psi = rE$, $\rho = r + r_g \ln(r/r_g - 1)$ is the tortoise coordinate, and the effective potential is

$$V = (1 - \frac{r_g}{r}) \left(\frac{2}{r^2} - \frac{3r_g}{r^3} + m^2 \right). \quad (10)$$

One can now set up an initial value problem by specifying the initial Ψ and $\dot{\Psi}$ and integrate eq.(9) forward in time. Since V is non-singular for all $-\infty < \rho < \infty$, an initially non-singular Ψ will remain so at all later times. Numerical experiments with various initial conditions confirm the anticipated decay of any initial electric field. Moreover, for small photon mass the decaying solutions asymptote to a mode that is best described by the intuitive picture of a quasi-stationary accretion of the screening charges onto the black hole: the *discharge mode*.

We formally define the discharge mode by postulating the exponential in time decay of the field $\Psi(t, \rho) = e^{-\gamma t} \Psi(\rho)$. The field is in-going at horizon and decreasing at spatial infinity:

$$\left(-\left(\frac{d}{d\rho}\right)^2 + \gamma^2 + V\right)\Psi = 0, \quad (11)$$

$$\frac{d\Psi}{d\rho} = -\gamma\Psi, \quad \rho \rightarrow -\infty, \quad (12)$$

$$\Psi \rightarrow 0, \quad \rho \rightarrow +\infty. \quad (13)$$

This eigenvalue problem is solved numerically in Appendix A, where we also show that our previous estimate of the discharge rate happens to be asymptotically exact: $\gamma = m^2 r_g$ in the limit $mr_g \rightarrow 0$.

We emphasize that the discharge mode is not a true long-time asymptote. For $t \gg (m^2 r_g)^{-1}$, the electric field will become oscillatory with an algebraic decay $t^{-5/6} \sin mt$, as shown in [2]. Nor does the discharge mode uniquely describe the time evolution of the electric field outside a charged star which collapses into a black hole. The evolution will depend on how exactly the collapse occurs, and the electric field will be given by a linear superposition of the decaying initial field and the field emitted by the charges moving into the black hole. If the charges move into the black hole fast enough, it seems reasonable to assume that the radiation will be mostly beamed into the black hole. The outside field is then probably dominated by the discharge mode. But in any case, the decay of the electric field cannot occur faster than the decay of the discharge mode. So to be precise, for small m , the intermediate-asymptotic decay of the near-hole electric field occurs in the discharge mode.

2.3 Discharge mode as a quasi-stationary solution

In the limit of small mr_g , the quasi-stationary character of the discharge mode becomes sharper. In this section we show that there exists a two-parameter family of quasi-stationary solutions characterized by the initial charge of the black hole q and the discharge rate \dot{q} . The previously defined discharge mode corresponds to the unique \dot{q}/q ratio for which the horizon is non-singular. The approximate method used in this section helps us extract analogous quasi-stationary solutions in massive gravity, where on the one hand, time-dependent simulations seem impossible, and on the other, $mr_g \ll 1$ is a reasonable assumption.

Let us return to the vacuum Einstein-Proca equations (1,2). The first observation to make is that when $mr_g \ll 1$, the expected discharge rate $\dot{q} \sim m^2 r_g q$ is so small that the

induced time-dependence of the electric field and the metric are negligibly small as compared to the r -dependence. The only effect of time-dependence is to allow $A_r \neq 0$. This can be seen from the expression for the screening charge current $J_s^\mu \equiv -m^2 A^\mu / 4\pi$, which gives

$$\dot{q} = m^2 \sqrt{-g} g^{rr} A_r \big|_{r=r_g}. \quad (14)$$

The second observation is that in the near-hole region we have $r \ll m^{-1}$, and as long as the fields are regular, the terms proportional to m in eqs.(1) and (2) are negligible. We therefore recover the time-independent Einstein-Maxwell theory, but now with an additional ‘gauge’ condition on A_μ

$$\partial_\mu (\sqrt{-g} g^{\mu\nu} A_\nu) = 0, \quad (15)$$

which is obtained by taking the divergence of eq.(1). This equation happens to be independent of m , and in our approximation it becomes an equation for A_r

$$\partial_r (\sqrt{-g} g^{rr} A_r) = 0. \quad (16)$$

This is just the condition of stationary flow on J_s^μ .

Assuming the Schwarzschild metric (7), we solve (16)

$$A_r = \frac{c}{r(r - r_g)}, \quad (17)$$

where c is a new integration constant related through eq.(14) to the discharge rate $c = -\dot{q}/m^2$. As for A_t , we can use the Einstein-Maxwell result $A_t = q/r$. Thus, we have a two-parameter family of quasi-stationary solutions characterized by the charge q and the discharge rate \dot{q} .

Requiring the horizon to be regular uniquely fixes the discharge rate. For instance, the regularity of the norm squared $g^{\mu\nu} A_\mu A_\nu = g^{tt} A_t^2 + g^{rr} A_r^2$ gives

$$\dot{q} = \pm m^2 r_g q, \quad (18)$$

which agrees with the previous calculation of the discharge mode in the limit $mr_g \rightarrow 0$. The sign is undetermined by our equations, which are invariant under the time-reversal, but physical considerations select the negative sign.²

3 Black hole disappearance in massive gravity

In this section, we first introduce Fierz-Pauli massive gravity, its nonlinear completions, and some of their relevant properties (§3.1). We next consider asymptotically flat spacetimes and argue that black holes must disappear (§3.2). In §3.3, we find quasi-stationary solutions, and show that unlike massive electrodynamics there is no unique disappearance rate. We next examine the implications of our findings in the actual problem of star collapse (§3.4). Finally in §3.5, we discuss the concept of global energy in massive gravity, and its failure in the presence of black holes.

²The above calculation can easily be generalized to the case where the back-reaction of charge on geometry is not negligible. The near-hole metric will then be the Reissner-Nordström solution and r_g in (18) will be replaced by r_+ , the radius of the outer horizon.

3.1 Fierz-Pauli massive gravity

By Fierz-Pauli massive gravity (FP), we denote a class of theories described by the Einstein-Hilbert action plus the (non-linear version of) Fierz-Pauli action:

$$S = S_{EH} + S_{FP}, \quad (19)$$

$$S_{EH} = -\frac{1}{2} \int d^4x \sqrt{-g} R, \quad (20)$$

$$S_{FP} = m^2 \int d^4x \sqrt{-g} U. \quad (21)$$

Here we have set $8\pi G = 1$, and U is a Lorentz-invariant potential defined using a flat reference metric η_{ab} . For small $h_{ab} = g_{ab} - \eta_{ab}$ it reduces to the Fierz-Pauli mass term [3]

$$U^{(2)} = \frac{1}{8} (h^2 - h_{ab}^2). \quad (22)$$

where $h = \eta^{ab} h_{ab}$. The Lorentz-invariance of U can be enforced by requiring that U be a symmetric function of the eigenvalues of the matrix $H_b^a = g^{ac} \eta_{bc}$.

The Fierz-Pauli term (22) is special because it is the unique quadratic expression that is linear in the perturbations of the ADM lapse function $\delta N = N - 1$. So as in general relativity, N remains a Lagrange multiplier and the theory describes five dynamical degrees of freedom at quadratic level. Rather obviously in the hindsight, this will no longer be the case beyond the quadratic level for a generic U , as Boulware and Deser pointed out [4]. This results in a sixth degree of freedom which they showed to be a ghost. This problem has been solved by de Rham, Gabadadze, and Tolley [5] who found a particular two-parameter family of potentials which propagates just five degrees of freedom at all orders. A concise representation of the family is [6]

$$U = \sum \lambda_a \lambda_b + \tilde{c}_2 \sum \lambda_a \lambda_b \lambda_c + \tilde{c}_3 \lambda_0 \lambda_1 \lambda_2 \lambda_3, \quad (23)$$

where the sums are over all all-distinct pairs and triples of indices, and λ_a are the four eigenvalues of the matrix

$$\delta_b^a - \sqrt{H_b^a}. \quad (24)$$

It is easy to understand why this two-parameter family, which we henceforth refer to as FP2, is special. Restricting to metrics with zero shift vector, the lapse appears only in $\lambda_0 = 1 - N^{-1}$. Apparently, the expression (23) is the only symmetric combination of λ_a , such that $\sqrt{-g} U_{FP2}$ has zero cosmological constant, and is linear in N . The full proof of the absence of Boulware-Deser ghost is given by Hassan and Rosen [7] (see also [8] for a different approach).³

As formulated above, the FP theory is not generally covariant. However, it can be made so by introducing 4 non-canonical scalar fields [14, 15, 16] to write H_b^a as

$$H_b^a = \eta_{bc} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^c. \quad (25)$$

³Evidently, having the right number of degrees of freedom does not protect FP2 against other pathologies such as strong coupling, instability, or superluminality of fluctuations when the background metric deviates from Minkowski (see, e.g., [9, 10, 11, 12]). The asymptotically flat solutions considered here are not expected to be exceptional [13].

In the so-called unitary gauge, one uses these scalar fields as coordinates, $X^a = \phi^a$, and recovers the original formulation of the FP theory. In the unitary gauge, the theory is still invariant under simultaneous reparametrization of g_{ab} and η_{ab} . For our purposes – studying spherically symmetrical stars and black holes – we transform to spherical coordinates. Then

$$\eta_{ab}dX^a dX^b = dT^2 - dR^2 - R^2 d\Omega^2. \quad (26)$$

If the space-time is spherically symmetric, the most general metric in this coordinate system is parametrized by four functions of T and R :

$$ds^2 = CdT^2 - 2DdTdR - AdR^2 - BR^2d\Omega^2. \quad (27)$$

A comment on our notation: in the following we will frequently switch to the time variable t as measured by asymptotic observers, and the circumference defined radius r (the Schwarzschild variables). However, the numerical indices $(0, 1, 2, 3)$ are exclusively used to denote unitary-gauge variables in the spherical coordinates in the order (T, R, θ, φ) .

3.2 Asymptotically flat spacetimes and the inevitable singularity of static black holes

As in massive electrodynamics, it is natural to first look for static star and black hole solutions in massive gravity. The procedure is briefly outlined in appendix B, and it was fully pursued in [17] with the conclusion that there exist acceptable star solutions but black holes are generically singular at horizon. However, again as in the case of massive electrodynamics, it is easy to exhibit the horizon singularity of static black holes without knowing the explicit solution. The horizon singularity now indicates that a star collapsing into a black hole must get rid of its very gravitational field and hence disappear.⁴

The horizon singularity can be shown as follows. Consider the most general static spherically symmetric vacuum solution of massive gravity. The unitary-gauge metric is given by (27). Requiring the space-time to be asymptotically flat then forces the metric to be diagonal [17], i.e. $g_{01} = D = 0$:

First note that the Ricci tensor of a time-independent metric of the form (27) satisfies the identity [18]

$$g_{01}R_{00} - g_{00}R_{01} = 0. \quad (28)$$

This imposes the following purely algebraic constraint on the unitary-gauge metric components via the vacuum massive gravity equations:

$$g_{01}T_{00} - g_{00}T_{01} = 0, \quad (29)$$

where T_{ab} is the stress-energy tensor of S_{FP} .

⁴Sergei Dubovsky pointed out that the horizon singularity of our static black hole solution [17] might signal accretion, like it does for fluids. If one tries to find a static fluid surrounding a black hole, one gets a singularity at horizon; in reality fluids are accreted.

On the other hand, for any potential U that is a symmetric function of the eigenvalues of the matrix $H_b^a = g^{ac}\eta_{cb}$, one can verify that $T_{01} = \kappa g_{01}$, where the proportionality coefficient κ is a non-singular function at $g_{01} = 0$ (see footnote 13). It follows that eq. (29) divides the solutions into two branches

- (i) $g_{01} = 0$,
- (ii) $T_{00} = \kappa g_{00}$.

Now consider large radii $R \rightarrow \infty$, where the metric perturbations $h_{ab} = g_{ab} - \eta_{ab}$ are small by the asymptotic flatness assumption. For any non-linear completion of the Fierz-Pauli mass term $h_{ab}^2 - h^2$, we then have

$$T_{01} = \frac{1}{2}m^2 h_{01}(1 + \mathcal{O}(h)), \quad (30)$$

giving $\kappa = m^2/2$ at large R . This excludes the branch (ii) as it leads to a finite asymptotic value for T_{00} . This branch is where the Schwarzschild-de Sitter solutions of [18, 19] are realized (see also [20] in that context).

On the other hand, whenever the unitary-gauge metric is diagonal, horizons will be physically singular in massive gravity [21]. This is because the inverse unitary-gauge metric components are scalar quantities in massive gravity: $g^{ab} = g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$. When g^{ab} is diagonal, its $\{00\}$ component will be singular at horizon and this singularity will be reflected in the action and the stress-energy tensor via $\lambda_0 = 1 - \sqrt{g^{00}}$. To have finite g^{ab} at horizon, one necessarily needs $g_{01} \neq 0$ which, by the above arguments, leads to time-dependence and non-zero energy flux T_0^1 .⁵

In view of black hole discharge in massive electrodynamics, the time-dependence of black holes in massive gravity is in fact naturally expected. The linearized field of a point source is known (and shown in §3.3.1) to exhibit the Yukawa decay $h_{ab} \propto \exp(-mr)$ in massive gravity. It follows that the invariant ADM mass of any localized system is zero. Therefore, the mass term, taken to the right hand side of the Einstein equation, can be thought of as the stress tensor of screening matter with negative energy that surrounds and degravitates gravitational sources. When a black hole forms, the screening matter flows inside and diminishes the black hole mass.⁶

The rate of this process is estimated by requiring the near horizon energy density $T_0^0 \sim -m^2$ accrete with the speed of light through an area of order r_g^2 :

$$\dot{r}_g \sim -m^2 r_g^2, \quad (32)$$

⁵Another way to see why $g_{01} \neq 0$ is a necessary condition for regularity of the horizon is to realize that when $g_{01} = 0$, the unitary-gauge time variable $T = \phi^0$ coincides with the proper time of the asymptotic observers t . But t is a singular variable at horizon and therefore $\phi^0 = t$ is singular [22]. This inevitable singularity relies just on the existence of a horizon, but if the Schwarzschild geometry is indeed recovered at short distances à la Vainshtein, we know how exactly t diverges at horizon: in terms of the advanced time v and circumference defined radius r which are regular variables

$$t = v - \rho = v - [r + r_g \ln(r/r_g - 1)]. \quad (31)$$

⁶The classical instability of black holes in massive gravity has been conjectured before by Gia Dvali [23].

where for a mass M black hole $r_g = M/4\pi$ in our units. Note that when $mr_g \ll 1$ the associated scale $\tau = 1/m^2 r_g$ is much longer than other length scales in the problem. We will use this fact to find approximate time-dependent solutions in the next section.

Assuming that the time-dependence is mild, of order (32), we can estimate the resulting value of the off-diagonal metric component g_{01} . Note first that the identity (28) is violated in the time-dependent case by terms of order

$$g_{01}R_{00} - g_{00}R_{01} \sim \partial_0 \partial_1 g_{00} \sim \dot{r}_g/r^2 \sim m^2 r_g^2/r^2, \quad (33)$$

where we used the estimate (32) in the last step. Next replace R_{ab} by T_{ab} via the Einstein equation, as we did to find the constraint (29), and use the same arguments to find

$$g_{01}(T_{00} - \kappa g_{00}) \sim m^2 r_g^2/r^2. \quad (34)$$

Since $g_{01} = 0$ as $r \rightarrow \infty$ and $\kappa \sim m^2$, we get $g_{01} \sim r_g^2/r^2$. Thus, we expect the diagonal static solution to be a good approximate solution everywhere except very close to the horizon where g_{01} becomes of order unity and makes the inverse metric g^{ab} finite. This expectation will be confirmed by the calculation of the next section.

3.3 Non-singular time-dependent black holes

It is relatively easy to find exact time-dependent solutions in the linear regime, but this is not the case at non-linear level. We therefore break the problem into three parts: first, we find the exact linear solutions and show that they are characterized by two parameters, the mass r_g and the disappearance rate \dot{r}_g . Then we use the $mr_g \ll 1$ approximation to find quasi-stationary near-hole solutions, again parametrized by the mass and rate. Finally, we match the linear and the near-hole solutions. We will see that unlike massive electrodynamics where all but a unique ratio \dot{q}/q lead to a singularity at the horizon, here all quasi-stationary solutions are regular. The analogous case of the accretion of superluminal fluids and possible explanations of this result are discussed in the following section §3.4.

3.3.1 Linear solutions

To study the linearized system, we parametrize the metric as

$$ds^2 = (1+c)dT^2 - 2d dTdR - (1+a)dR^2 - (1+b)R^2 d\Omega^2, \quad (35)$$

where a, b, c, d are infinitesimal functions of T and R . The linearized Einstein equation in Cartesian coordinates reads

$$\square h_{ab} - \partial_a \partial_c h_b^c - \partial_b \partial_c h_a^c + \partial_a \partial_b h - \eta_{ab} \square h + \eta_{ab} \partial_c \partial_d h^{cd} = m^2 (\eta_{ab} h - h_{ab}), \quad (36)$$

and its divergence yields the condition $\partial_a h_b^a = \partial_b h$. Since the equations are first order in perturbations h_{ab} , we can replace in them (T, R) with (t, r) which coincide at zeroth order. Moreover, the solutions can be expanded in the exponential basis

$$a, b, c, d \propto e^{-\lambda m t}. \quad (37)$$

The system of equations can then be brought into the form

$$a'r + 2a - 2b = \frac{\lambda^2}{1 - 2\lambda^2} m^2 r^2 (a - 2\lambda^2 b), \quad (38)$$

$$b'r + b - a = \frac{1}{2(1 - 2\lambda^2)} m^2 r^2 (a - 2\lambda^2 b), \quad (39)$$

where $' \equiv d/dr$, and c and d are given in terms of a, b by

$$c'r = 2(1 + \lambda^2)(a - b - b'r), \quad (40)$$

$$d = \frac{2\lambda}{mr}(a - b - b'r). \quad (41)$$

Thus, for any fixed rate $|\lambda| < 1/\sqrt{2}$ there exists a one-parameter family of decaying as a function of r solutions. The parameter characterizes the mass of the gravitating body.

Note that in agreement with the arguments of the previous section, g_{01} vanishes in the static limit $\lambda \rightarrow 0$. For an accretion rate of order (32), we have $\lambda \equiv \alpha m r_g \ll 1$ (where we introduced α as a so far undetermined order-unity parameter characterizing the rate \dot{r}_g). The solution can therefore be approximated by the $\lambda = 0$ vDVZ solution [24]:

$$c = -\frac{4r_g}{3r} e^{-mr}, \quad a = -\frac{4r_g}{3m^2 r^3} e^{-mr} (1 + mr), \quad b = \frac{2r_g}{3m^2 r^3} e^{-mr} (1 + mr + m^2 r^2), \quad (42)$$

except that r_g adiabatically changes with time as $r_g(t) \simeq r_g(0) \exp(-\alpha m^2 r_g t)$, and

$$d = \frac{4\alpha r_g^2}{3r^2} e^{-mr} (1 + mr). \quad (43)$$

We see that, apart from allowing small non-zero g_{01} , the time-dependence is inconsequential in the long-distance linear field.⁷ In FP2, this linear solution is valid for $r \gg r_V = (r_g/m^2)^{1/3}$.

3.3.2 Near-hole solutions

Similar to massive electrodynamics, we can develop an approximate method to find the short distance ($r \ll r_V$) quasi-stationary solutions in the limit $mr_g \ll 1$. This approximation is in fact a generalization of the Vainshtein's original idea [27], which we now review and extend to our case.

Aiming for a static solution, Vainshtein started from a diagonal metric ansatz

$$ds^2 = C dT^2 - A dR^2 - B R^2 d\Omega^2, \quad (44)$$

⁷We also see that in the context of massive gravity the no-hair theorem of [25] and in general the singularity of static black holes cannot be interpreted as a dramatic difference between the field of stars and black holes. This can be understood from the fact that the linear static field of stars and black holes is uniquely determined by the mass r_g . To lose its hair, the black hole must actually lose its entire mass, which is done only very slowly. In particular, we think that contrary to the claims of [26], the response of stars and black holes of the same mass to the field of distant objects is almost indistinguishable.

where A, B, C are functions of R , and are determined using the Einstein equations

$$G^\mu_\nu = T^\mu_\nu. \quad (45)$$

He noticed that T^μ_ν is proportional to m^2 and therefore can be ignored at short radii, as long as the metric coefficients remain finite. Therefore, the mass-less Einstein theory of gravity is recovered in this limit, implying that the metric (44) must be a reparametrization of the Schwarzschild metric

$$ds^2 = (1 - \frac{r_g}{r})dt^2 - (1 - \frac{r_g}{r})^{-1}dr^2 - r^2d\Omega^2. \quad (46)$$

The reparametrization is determined from the covariant divergence of eq.(45):

$$\nabla_\mu T^\mu_\nu = 0, \quad (47)$$

which can be thought of as a gauge condition on the metric (44). Note that m drops out of this equation, so it is a non-trivial constraint even in the zero-graviton-mass limit.

Requirement of asymptotic flatness ($g_{ab} = \eta_{ab}$ as $R \rightarrow \infty$) then fixes $T = t$, and the only non-trivial component of eq.(47) (the $\nu = 1$ component) serves as an equation for $R(r)$, in terms of which A, B, C are given by

$$C = 1 - \frac{r_g}{r}, \quad A = 1/(CR'^2), \quad B = r^2/R^2. \quad (48)$$

Vainshtein realized that while there is no such reparametrization of the Schwarzschild metric at the linearized level (the well-known vDVZ discontinuity), linearization becomes inadequate at $r \sim r_V$. He showed that at the non-linear level there exists a finite solution valid for $r_g \ll r \ll r_V$, which a posteriori justifies neglecting T^μ_ν from the Einstein equation. One still needs to check whether this solution matches the vDVZ solution for $r \gg r_V$, where the mass term cannot be ignored anymore. This is a non-trivial check, but there exists a sub-family of FP2 for which the answer is positive [17, 28] (see [29] for earlier works, and [30] for a related discussion).

Returning to the time-dependent problem, we expect that, as in massive electrodynamics where for small mr_g the only relevant effect of the time-dependence was to excite A_r , here the time-dependence excites g_{01} . Once this is taken into account by using the metric ansatz

$$ds^2 = CdT^2 - 2DdTdR - AdR^2 - BR^2d\Omega^2, \quad (49)$$

we can ignore time derivatives (quasi-stationary approximation) and repeat Vainshtein's procedure. Neglecting the stress tensor of the FP action from the Einstein equation implies that the metric is a special reparametrization of the Schwarzschild metric which satisfies the gauge condition $\nabla_\mu T^\mu_\nu = 0$.

In the quasi-stationary approximation, the metric (49) can be diagonalized and expressed in terms of t and r by the coordinate transformations

$$R = B^{-1/2}r, \quad (50)$$

$$T = t + \int \frac{D}{C} \gamma dr, \quad (51)$$

where $\gamma \equiv dR/dr = B^{-1/2}(1 - B'r/2B)$. In this coordinate system the metric looks like

$$ds^2 = Cdt^2 - \gamma^2(A + D^2/C)dr^2 - r^2d\Omega^2. \quad (52)$$

Requiring this to match the Schwarzschild metric (46) gives

$$C = 1 - r_g/r, \quad A = C^{-1}(\gamma^{-2} - D^2). \quad (53)$$

Hence, there are two unknowns D and B , characterizing the gauge transformation from the unitary frame to the Schwarzschild frame. The task is to solve for them using the $\nu = 0, 1$ components of $\nabla_\mu T_\nu^\mu = 0$ on a fixed Schwarzschild geometry, and subject to appropriate boundary conditions. The resulting quasi-stationary solutions are expected to be parametrized by the mass r_g and the disappearance rate $\dot{r}_g = \alpha m^2 r_g^2$ (which will be the integration constant of the $\nu = 0$ equation). One must then ask for what values of α the solution is regular at horizon.⁸

Note that the above procedure is equivalent to solving the equations of motion for the scalar fields ϕ^a on a fixed Schwarzschild background (as one usually treats accretion problems [31]). By spherical symmetry $\phi^i = Rn^i = B^{-1/2}x^i$, and the stationary approximation corresponds to looking for solutions of the form $\phi^0 = t + \varphi(r)$ [which is related to B and D by eq.(51)]. This is why there are only two non-trivial equations; they are related to the stress-energy conservation via the Bianchi identity

$$\nabla_\nu T_\mu^\nu = -\frac{1}{\sqrt{-g}}\partial_\mu \phi^a \frac{\delta S}{\delta \phi^a}, \quad (54)$$

which is particularly simple in the unitary gauge where $\partial_b \phi^a = \delta_b^a$. Neglecting the back-reaction on geometry is justified for small graviton mass if we find a regular solution.

The details of the calculation for FP2 is given in appendix C. In summary, we obtain an algebraic equation for $\lambda_2 = 1 - B^{-1/2}$:

$$\beta(1 - \frac{3r_g}{4r}) + (c_2 + c_3\lambda_2)(1 + (1 - \lambda_2)C) = \left(C + \frac{\alpha^2 r_g^4}{\beta^2 r^4}\right)^{1/2} (1 + c_2\lambda_2 + 2(c_2 + c_3\lambda_2)), \quad (55)$$

where $c_{2,3}$ are related to the parameters of FP2 action (23), and $\beta \equiv 1 + 2c_2\lambda_2 + c_3\lambda_2^2$ is a positive function when $c_2^2 \leq c_3$, the range where stable numerical time-independent solutions exist. Given α and r_g , eq.(55) is solved for λ_2 , using which D is given by⁹

$$D = \frac{\alpha r_g^2}{\beta r^2}(\gamma^{-1} + C) \left(C + \frac{\alpha^2 r_g^4}{\beta^2 r^4}\right)^{-1/2}. \quad (56)$$

⁸In the language of footnote 5, one seeks solutions on which ϕ^0 interpolates between $\phi^0 = t$ at $r = \infty$ and $\phi^0 = t + \rho$ at $r = r_g$ [22].

⁹The uniqueness of the solution once r_g and \dot{r}_g are fixed is a sign of the absence of Boulware-Deser ghost in FP2. Among the five degrees of freedom of a massive graviton only the scalar mode is dynamical in the spherically symmetric problem, and can participate in the accretion (analogous to the longitudinal polarization of the massive photon and the sound mode of fluids). After the large- r asymptotic condition is fixed by the mass r_g (charge q for photon, asymptotic density for fluids), only one additional integration constant is needed to fully parametrize the stationary accretion. In generic FP, on the other hand, the analogue of equation (55) for B is of the second order, which requires two extra integration constants. These correspond to the emergence of a second scalar degree of freedom, which is the Boulware-Deser ghost.

Now we can estimate how much the time-dependent solutions deviate from the static solution. Assuming $\alpha \sim 1$, equation (56) gives a small off-diagonal metric component at $r \gg r_g$:

$$D \simeq \frac{\alpha r_g^2 (1 + \gamma^{-1})}{\beta r^2}, \quad (57)$$

and eq.(55) reduces to its static $\alpha = 0$ limit, giving $\lambda_2 = 1/\sqrt{c_3}$. Therefore, as anticipated in §3.2 the effects of time-dependence are felt merely within a distant of a few r_g from the horizon, beyond which the static solution of [17] remains accurate. In particular, matching to the linearized solution (studied in appendix D) does not impose any constraint. This should not be very surprising in view of the fluid accretion problem. The critical radius that determines the accretion rate for fluids is close to the horizon for relativistic fluids. This is also what one expects from the example of massive electrodynamics where regularity of the horizon itself determines the accretion rate.

What is the actual rate of accretion \dot{r}_g ? In massive electrodynamics, the discharge solution has been selected from the two-parametric (q, \dot{q}) family of quasi-stationary solutions by the horizon non-singularity. If massive gravity was fully analogous to electrodynamics, the rate \dot{r}_g would also be uniquely determined by the horizon non-singularity. But this does not actually happen. For $r \rightarrow r_g$ (and $C \rightarrow 0$), eq.(56) gives

$$D = \epsilon \gamma^{-1} + \epsilon C \left(1 - \frac{\beta^2 r^4}{2\alpha^2 r_g^4 \gamma} \right), \quad \epsilon = \frac{\alpha}{|\alpha|}, \quad (58)$$

and upon substitution in (51) gives $\phi^0 = T = t + \epsilon \rho + \mathcal{O}(\alpha^{-2})$, which is regular at black hole horizon for any positive α (see footnote 8).¹⁰ So in contrast to massive electrodynamics, any non-zero rate seems to result in a regular solution in massive gravity. This behavior, though peculiar, is not unprecedented. It is known [32] (and reviewed in appendix F) that the accretion rate of fluids with superluminal speed of sound onto black holes is not unique. In the next section, we examine the implications of this finding for the actual process of star collapse, where we argue that the rate is determined by the history of collapse.

3.4 The fate of collapsing stars

In classical physics, an outside observer never sees a fully formed black hole – collapsing stars are stuck forever at their gravitational radius with redshift increasing at their surface. In this section, we study black hole accretion from this perspective which, besides being more realistic, provides a useful thought laboratory to examine our analytic results. We first consider the accretion of fluids onto collapsing stars. We then give a related interpretation of black hole discharge in massive electrodynamics, and finally, turn to the problem of collapsing stars in massive gravity.

¹⁰If one only requires the regularity of g^{ab} , then negative values of α are also admissible. In that case, the black hole excretes negative energy ‘aether’ and grows.

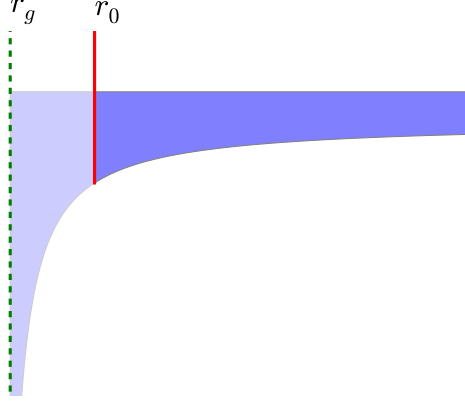


Figure 1: The density of static fluids (represented by the depth) always diverges at horizon. However, the total amount of fluid mass that can be stored at rest between the star surface and the horizon is infinite for $c_s^2 \leq 1$, while finite for $c_s^2 > 1$.

3.4.1 Fluids

Ideal fluids are characterized by their density, pressure, and velocity fields, respectively ε , p , and u^μ , and a certain equation of state that relates ε and p . The fluid stress-energy tensor is

$$\mathcal{T}_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - pg_{\mu\nu}. \quad (59)$$

Consider a spherically symmetric static solution on the Schwarzschild metric (we ignore the fluid back-reaction on geometry). The $\nu = 1$ component of the stress-energy conservation, $\nabla_\mu \mathcal{T}_\nu^\mu = 0$, gives

$$p' = \frac{-r_g}{2r(r - r_g)}(\varepsilon + p). \quad (60)$$

Assuming a simple $p = c_s^2 \varepsilon$ equation of state, we get the density field

$$\varepsilon = \varepsilon_0 \left(1 - \frac{r_g}{r}\right)^{-(1+c_s^{-2})/2}, \quad (61)$$

where ε_0 is the density at spatial infinity. We see that for subluminal and superluminal fluids alike the density and pressure diverge at $r = r_g$, reflecting the fact that an infinite force is needed to hold a test particle at rest above the horizon.

However the two cases, subluminal and superluminal, are known to be very different when one considers the black hole accretion. The rate of accretion is uniquely determined by the critical Bondi solution in the subluminal case, $c_s^2 \leq 1$, because accreting solutions with slower rates are singular at horizon, while solutions with faster rates do not exist. On the other hand, in the superluminal case, when $c_s^2 > 1$, the critical accretion rate does not exist and all accreting solutions are regular at horizon (see appendix F).

This difference is ultimately due to the different amounts of *static* fluid mass which can be stored near the black hole horizon. On Schwarzschild metric, there is a well-defined

notion of fluid mass since the $\nu = 0$ component of the stress-energy conservation becomes $\partial_\mu(\sqrt{-g}\mathcal{T}_0^\mu) = 0$ and implies the conservation of

$$M = \int d^3x \sqrt{-g} \mathcal{T}_0^0. \quad (62)$$

At spatial infinity where the fluid is at rest and the metric is flat this reduces to $\int d^3\mathbf{r} \varepsilon$. Hence, M is the fluid energy as measured by asymptotic observers. Now consider a high redshift star of radius r_0 as a model for the black hole. Using the static solution (61), it is easy to see that when $c_s^2 < 1$ the fluid mass near the star surface diverges as $r_0 \rightarrow r_g$, while it remains finite for $c_s^2 > 1$ (figure 1).

Imagine next initiating the fluid accretion by contracting the star (classically, r_0 will pass the horizon only at $t = \infty$, no matter how fast the contraction takes place, so the accretion problem can be fully addressed in this framework). When $c_s^2 < 1$, the above-mentioned divergence of near-horizon fluid mass makes it possible to sustain an infinitesimal rate of steady accretion by adiabatically contracting the star. The pressure and density in this regime are then approximately given by the singular static solution. This explains the singularity of those sub-critical accreting solutions. In contrast, because of the finiteness of near-horizon M in the $c_s^2 > 1$ case, it is impossible to sustain any steady accretion by adiabatic contraction of the star. Consequently, the accreting solutions of superluminal fluids can never be approximated by the singular static solution near the horizon.

Without an explicit calculation of fluid accretion we cannot proceed much further. So in the following, besides filling in some of the details of the above picture, we use the results of appendix F to provide a more comprehensive description.

Let us define $x = r - r_g$ in terms of which the $t - r$ part of the near-horizon Schwarzschild geometry is given by the Rindler metric:

$$ds^2 = r_g^{-1} x dt^2 - r_g x^{-1} dx^2. \quad (63)$$

For a radially moving time-like curve near the horizon, we have

$$\dot{x}^2 = r_g^{-2} x^2 (1 - r_g^{-1} x e^{-2}), \quad (64)$$

where the overdot denotes t -derivative, and $e \equiv u_0$ is the energy per unit rest mass as measured from infinity. e is constant for free-falling observers, but it vanishes as $e = (x/r_g)^{1/2}$ for static ones. Thus, the maximal value of $|\dot{x}|$ near the horizon is $|\dot{x}|_{\max} = x/r_g$.

As a function of the position of the star surface $x_0 = r_0 - r_g$, the mass of static fluid with density (61) that resides near the surface scales as

$$x_0^{(1-c_s^{-2})/2}. \quad (65)$$

When $c_s^2 < 1$, this diverges in the limit $x_0 \rightarrow 0$ and, as explained above, infinitesimal rates of steady accretion can be obtained by adiabatically decreasing x_0 :

$$\dot{M} = -4\pi r_g^2 \varepsilon \dot{x}_0, \quad (66)$$

with ε given by (61). As $|\dot{x}_0|$ is increased to the point that this rate exceeds the critical Bondi rate, the fluid dynamics decouples from the motion of the surface and settles to the regular Bondi solution.

On the other hand, when $c_s^2 > 1$ substituting the maximal value $|\dot{x}_0|_{\max} = x_0/r_g$ in eq. (66) gives zero as $x_0 \rightarrow 0$. Thus, similar adiabatic contractions do not correspond to any steady accretion for superluminal fluids. This result may seem rather counterintuitive as it suggests that steady accretion is impossible when $c_s^2 > 1$. This is of course not the case. Consider again the near horizon contribution to M for a possibly moving fluid:

$$M(x_0) \sim 4\pi r_g^2 \int_{x_0} dx [(\varepsilon + p)r_g e^2 x^{-1} - p], \quad (67)$$

where the expression in the square brackets is \mathcal{T}_0^0 for a fluid with $u_0 = e$. Substituting the infinitely redshifted $e = (x/r_g)^{1/2}$ for a static fluid, we see that the density of superluminal fluids is not singular enough and the integral converges as $x_0 \rightarrow 0$. However, if the same fluid is allowed to fall in, with $e \neq 0$ at $x = 0$, even with a finite ε and p an infinite amount of mass can be deposited near the horizon. The flux

$$\mathcal{T}_0^1 = (\varepsilon + p)r_g e^2 x^{-1} \dot{x}, \quad (68)$$

would then be non-zero. The explicit solution of appendix F shows that any non-zero accretion rate of superluminal fluids can indeed be obtained by a specific choice of e at horizon. So different free-fall ($e = \text{const.}$) motions of the star surface at horizon lead to different rates of accretion, implying that unlike subluminal fluids the accretion of a superluminal fluid never decouples from the star collapse. Moreover, since even for infinitesimal accretion rates the fluid is free-falling at horizon, it is expected that the pressure remains finite there.

Given that at any finite t all of the accreted matter is still above the horizon, one may wonder what happens if the contraction of the star is paused after a long period Δt of steady accretion. Clearly the static solution must be recovered since the star surface is at a finite redshift. The finiteness of the near horizon static fluid mass when $c_s^2 > 1$ then implies that, for large enough Δt , the star must excrete most of the fluid that has been accumulated near the surface back to larger radii.

Finally, when $c_s^2 = 1$ the total mass of static fluid adjacent to the surface diverges logarithmically with x_0 . Therefore, in the adiabatic approximation (66) even infinitesimal rates require very fast motion ($|\dot{x}_0| \propto x_0$, although with $e \propto x^{1/2}$) which makes this approximation unreliable. A more detailed analysis is needed to show the singularity of sub-critical accreting solutions in this case.

3.4.2 Massive electrodynamics

Now let us interpret the black hole discharge in massive electrodynamics by considering a charged shell of radius r_0 approaching its gravitational radius r_g . We first need to determine the relation between the asymptotic field $A_t = qe^{-mr}/r$ and the actual charge Q on the shell as a function of r_0 . This can be obtained from the discontinuity of the electric field at the surface in the following way (a more detailed analysis can be found in [33]):

Ignoring back-reaction on geometry, and substituting $A_t = q\chi/r$ in the source-free Proca equation, we get for $r > r_0$

$$\chi'' = m^2(1 - \frac{r_g}{r})^{-1}\chi. \quad (69)$$

For small mr_g , the decaying solution at infinity $\chi = e^{-mr}$ goes into

$$\chi = 1 + m^2 r_g x \ln x \quad (70)$$

near the horizon. On the other hand, inside the shell $r < r_0$, where $g_{rr} = -1$ and $g_{tt} \simeq x_0/r_g = \text{const.}$, we have

$$\chi'' = m^2\chi. \quad (71)$$

Imposing regularity condition at the center and matching to the outside solution we get for $r < r_0$

$$A_t = \frac{q}{r_0} \frac{\sinh mr}{mr}. \quad (72)$$

The charge on the shell is then given by the discontinuity of proper electric field at r_0

$$Q = \sqrt{-g} g^{rr} g^{tt} F_{rt} \Big|_{r_0^-}^{r_0^+} \simeq qm^2 r_g^2 \left[\ln(r_g/x_0) + \frac{1}{3}(r_g/x_0)^{1/2} \right], \quad (73)$$

where we have kept both the contributions from outside and inside although the former is sub-dominant in the $x_0 \rightarrow 0$ limit.

To have a fixed asymptotic field (characterized by q), it is seen from eq.(73) that the total charge on the shell Q must be increased indefinitely as the surface redshift is taken to infinity. This is due to the divergence of the total amount of screening charge both inside and outside of the shell, respectively as $x_0^{-1/2}$ and $\ln x_0$. In the realistic problem, as a star of fixed charge Q contracts to smaller radii, its asymptotic field diminishes since the surrounding screening charges accrete into the now deeper potential well around and inside the star.

As in the case of subluminal fluids the divergence of total static charge in the limit $x_0 \rightarrow 0$ allows us to reproduce small rates of accretion by slowly contracting the shell:

$$\dot{q} = -\frac{\partial Q}{\partial x_0} \dot{x}_0 \simeq \frac{1}{6} q m^2 r_g^{5/2} x_0^{-3/2} \dot{x}_0. \quad (74)$$

However, here the screening charges do penetrate inside the shell and the slow motion of the surface does not imply that of the charges. In fact, the logarithmic divergence of charge density outside the shell makes this case analogous to fluids with $c_s = 1$, and a detailed analysis is needed to show why discharge rates of less than $m^2 r_g$ lead to a singular horizon. As the shell collapses faster, the outside field eventually decouples from the motion of the shell and decays via the discharge mode.

3.4.3 Massive gravity

For static black holes in massive gravity, we show in appendix B.1 that while pressure usually diverges at the horizon, the energy density always remains finite in FP2, and so does the total amount of static energy that can be stored outside the horizon. Therefore, as in the case of superluminal fluids, to have any non-zero rate of accretion the surrounding screening matter should freely fall at horizon and hence the pressure is expected to remain finite. This seems to be the reason for the non-uniqueness of the disappearance rate. The rate will then depend on the motion of the collapsing star according to one of the following scenarios:

i) If the total amount of negative energy inside static stars diverges with their redshift (similar to the divergence of total screening charge in the interior of the charged shell in massive electrodynamics), then by slowly contracting the star to higher redshifts we obtain arbitrary rates of accretion. This implies that the gravitational radius r_g (which is approximately the actual radius of the star) shrinks as the redshift increases, and the perceived mass of the star decreases. In this case if the process of collapse is paused at some point the result will be a static star of smaller gravitational radius.

ii) Otherwise, as in the case of impenetrable stars surrounded by superluminal fluids, non-zero accretion rates are obtained when the surface freely falls which allows for the accumulation of an arbitrary amount of free-falling screening matter. As before, the gravitational radius (and the real radius) of the system shrinks in this process. However, if we decide to pause the collapse, the excess of screening matter must return to larger radii, resulting in an excreting star with ever increasing surface redshift.

We were unable to find high redshift static star solutions in FP2 to decide between the two scenarios. Nevertheless, in both cases as the star shrinks to very small radii the quantum mechanical effects become important and the above description in terms of stars breaks down. The resulting small mass black hole will probably evaporate via Hawking radiation.

3.5 A substitute for the ADM mass

Let us conclude by a discussion of global energy in asymptotically flat solutions of massive gravity. We have already seen that the invariant ADM mass of a localized system trivially vanishes in massive gravity because of the Yukawa screening. Equivalently, if one defines an ordinarily conserved pseudo-tensor of stress and energy, say \mathfrak{T}_μ^ν , the conserved total energy-momentum four-vector of the system which is obtained by integrating \mathfrak{T}_μ^0 over the whole space is always zero. Intuitively, the negative contribution of S_{FP} cancels the energy and momentum of the ‘matter content’, by which we mean everything except S_{FP} . Of course, on a curved space-time these different contributions are not separately conserved, and seemingly, nothing forbids the growth or disappearance of the matter content at the expense of the growth and disappearance of the FP content. However, there exist four Noether charges, associated with four global symmetries of the Fierz-Pauli theory, which as we will see are closely related to the negative energy-momentum of S_{FP} . As such, they are also a well-defined measure of the matter-content of the theory. The conservation of these charges, then, constrain processes that involve stars, but they can be violated in the presence of

black holes.¹¹

The prescription (25) to make S_{FP} covariant, also introduces four global symmetries under constant shifts of the scalar fields ϕ^a . Therefore, there are four conserved Noether currents obtained by varying S_{FP} with respect to $e_\mu^a = \partial_\mu \phi^a$

$$J_a^\mu = \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta e_\mu^a}, \quad (75)$$

and associated to them, there will be four conserved charges Q_a . Moreover, the form of the Lagrangian $U(g^{\mu\nu} e_\mu^a e_\nu^b \eta_{cb})$ allows us to relate J_a^μ to the stress-tensor of S_{FP}

$$J_a^\mu = e_a^\nu (T_\nu^\mu + m^2 \delta_\nu^\mu U), \quad (76)$$

where $T_{\mu\nu} = 2(-g)^{-1/2} \delta S_{FP} / \delta g^{\mu\nu}$, and e_μ^a is the inverse of e_μ^a . In the unitary gauge $e_b^a = \delta_b^a$, and therefore, e_μ^a is invertible as long as this gauge exists. Consider now a localized material system whose center of mass is at rest; Q_0 has the following properties:

i) When the matter distribution is so dilute that the perturbations of the unitary-gauge metric are infinitesimal ($h_{ab} \ll 1$), we have $J_b^a = T_b^a + \mathcal{O}(h_{ab}^2)$. It follows that $Q_0 = -M$, where $M = \int d^3\mathbf{r} \mathcal{T}_0^0$ is the total energy of the matter distribution. This can be seen from the fact that when linearization is possible, \mathfrak{T}_μ^ν is simply the total stress-energy tensor appearing on the right hand side of the Einstein equation, so we have

$$M_{\text{ADM}} = \int d^3\mathbf{r} \mathfrak{T}_0^0 = \int d^3\mathbf{r} (T_0^0 + \mathcal{T}_0^0) = 0. \quad (77)$$

The condition $h_{ab} \ll 1$ is satisfied as long as $G\mathcal{T}_0^0 \ll m^2$ and $GM^{2/3}\mathcal{T}_0^{1/3} \ll 1$.

ii) Inside the Vainshtein region, the gravitational field of a star is the same as in the Einstein theory. For a star at finite redshift one can show that $M = -Q_0$, where $M = 4\pi r_g$ is the Schwarzschild mass of the star as determined from its Einsteinian field:

Consider a sphere of radius r_1 where $r_g \ll r_1 \ll r_V = (r_g/m^2)^{1/3}$. The space-time is already nearly flat at r_1 , so the ADM mass inside the sphere is well-defined and equal to M . The ADM mass inside a much larger sphere of radius $r \rightarrow \infty$ is zero, so we must have

$$\int_{r_1}^{\infty} d^3\mathbf{r} T_0^0 = -M. \quad (78)$$

In appendix E we show that

$$\int_{r_1 \gg r_g}^{\infty} d^3\mathbf{r} J_0^0 = \int_{r_1 \gg r_g}^{\infty} d^3\mathbf{r} T_0^0. \quad (79)$$

Moreover, the assumption of finite redshift ensures that $J_0^0 = \mathcal{O}(m^2)$ and the contribution of the region $r < r_1$ to Q_0 is negligible, because $J_0^0 r_1^3 \ll J_0^0 r_V^3 = \mathcal{O}(r_g)$. This fact together with equations (78) and (79), implies $M = -Q_0$.

¹¹We thank Gregory Gabadadze for discussions and collaboration that lead to the identification of these charges.

iii) If the redshift of the star diverges as inverse powers of mr_g , large amounts of screening matter (presumably with negative energy) can be accumulated in the region $r \sim r_g$, in which case the Schwarzschild mass of the star decreases and deviates from Q_0 .

iv) When a black hole forms and evaporates conservation of global charges such as Q_0 can be violated. Therefore, the vanishing of the asymptotically defined M_{ADM} in massive gravity allows the post black hole matter content to be less than the initial one.

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A Discharge mode in massive electrodynamics

We numerically solve the eigenvalue problem

$$\left(-\left(\frac{d}{d\rho}\right)^2 + \gamma^2 + V\right)\Psi = 0, \quad \frac{d\Psi}{d\rho} = -\gamma\Psi|_{-\infty}, \quad \Psi = 0|_{+\infty}. \quad (\text{A.1})$$

We take $\Psi(\rho_1) = e^{-\gamma\rho_1}$, $\frac{d\Psi}{d\rho}(\rho_1) = -\gamma e^{-\gamma\rho_1}$, where ρ_1 is an arbitrary (but negative and large in absolute value) number. We then integrate eq.(A.1) from ρ_1 to large positive ρ . We choose γ so as to get $\Psi \rightarrow 0$, $\rho \rightarrow +\infty$. This gives the decay rates γ .

The resulting decay rate, for $mr_g < 0.5$, to better than 3% accuracy, is given by a fitting formula

$$\gamma \approx \frac{m^2 r_g}{1 + mr_g}. \quad (\text{A.2})$$

The asymptotic

$$\gamma \rightarrow m^2 r_g, \quad mr_g \rightarrow 0 \quad (\text{A.3})$$

can be shown to be exact. One first calculates the (singular at horizon) stationary electric field from the Proca equation (8)

$$\frac{d}{dr} \left(\left(1 - \frac{r_g}{r}\right) \frac{1}{r^2} \frac{d}{dr} (r^2 E) \right) = m^2 E. \quad (\text{A.4})$$

It follows that

$$\frac{d\Psi}{d\rho} + \left(1 - \frac{r_g}{r}\right) \frac{1}{r} \Psi = -m^2 r \int_r^\infty dr E. \quad (\text{A.5})$$

For small m , the right-hand side of eq.(A.5) can be approximated by the zero-mass solution $E = q/r^2$, giving

$$\frac{d\Psi}{d\rho} + \left(1 - \frac{r_g}{r}\right) \frac{1}{r} \Psi = -m^2 q. \quad (\text{A.6})$$

For r close to the horizon, $r - r_g \ll r_g$, we get

$$\frac{d\Psi}{d\rho} = -m^2 q. \quad (\text{A.7})$$

The decay mode, for $r - r_g \ll r_g$, is $\Psi = Ce^{-\gamma\rho}$ where C is some constant, so that

$$\frac{d\Psi}{d\rho} = -\gamma\Psi. \quad (\text{A.8})$$

On the other hand, for negative ρ in the interval $1 \ll |\rho| \ll \gamma^{-1}$, the decay mode and the zero-mass static mode are approximately equal to the Coulomb field at horizon, $\Psi \approx q/r_g$, and comparing eqs.(A.7, A.8) we get $\gamma = m^2 r_g$.

B Static spherically symmetric field

In this appendix we outline the method used to find the static solution and a few relevant results of [17]. Starting from the unitary gauge, the asymptotically flat metric can be written as

$$ds^2 = e^\nu dt^2 - e^{\tilde{\lambda}} dR^2 - R^2 e^\mu d\Omega^2. \quad (\text{B.1})$$

We then change the radial coordinate R to the circumference defined r

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\Omega^2. \quad (\text{B.2})$$

The scalar fields in the new coordinates become

$$\phi^0 = t, \quad \phi^i = r e^{-\mu/2} n^i, \quad (\text{B.3})$$

where n^i is the unit radial vector.

As independent vacuum equations we use two Einstein equations and the stress-energy conservation

$$G_0^0 = r^{-2}(1 - e^{-\lambda}) + r^{-1}e^{-\lambda}\lambda' = T_0^0, \quad (\text{B.4})$$

$$G_1^1 = r^{-2}(1 - e^{-\lambda}) - r^{-1}e^{-\lambda}\nu' = T_1^1, \quad (\text{B.5})$$

$$T_1^{1'} = \frac{1}{2}\nu'(T_0^0 - T_1^1) + \frac{2}{r}(T_2^2 - T_1^1), \quad (\text{B.6})$$

where prime denotes the r -derivative. Inside stars one adds the matter stress-energy tensor \mathcal{T}_μ^ν to the right hand side of eqs.(B.4,B.5), and separately imposes its conservation $\nabla_\nu \mathcal{T}_\mu^\nu = 0$.

The stress-energy tensor of FP2 is derived from eq.(23):

$$T_0^0 = -m^2(\lambda_1 + 2\lambda_2 + c_2(2\lambda_1\lambda_2 + \lambda_2^2) + c_3\lambda_1\lambda_2^2), \quad (\text{B.7})$$

$$T_1^1 = -m^2(\lambda_0 + 2\lambda_2 + c_2(2\lambda_0\lambda_2 + \lambda_2^2) + c_3\lambda_0\lambda_2^2), \quad (\text{B.8})$$

$$T_2^2 = -m^2(\lambda_0 + \lambda_1 + \lambda_2 + c_2(\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2) + c_3\lambda_0\lambda_1\lambda_2),$$

with

$$c_2 = 1 + \tilde{c}_2, \quad c_3 = \tilde{c}_2 + \tilde{c}_3, \quad (\text{B.9})$$

and

$$\lambda_0 = 1 - e^{-\nu/2}, \quad (\text{B.10})$$

$$\lambda_1 = 1 - e^{-\tilde{\lambda}/2} = 1 - e^{-\lambda/2}(1 - \lambda_2 - r\lambda'_2), \quad (\text{B.11})$$

$$\lambda_2 = 1 - e^{-\mu/2}. \quad (\text{B.12})$$

Using expressions (B.7,B.8,B.9) and expression (B.10,B.11) in eqs.(B.4,B.5,B.6) gives the following system of three equations for three unknowns ν , λ , and λ_2 :

$$1 - e^{-\lambda} + re^{-\lambda}\lambda' = -m^2r^2(2\lambda_2 + c_2\lambda_2^2 + (1 + 2c_2\lambda_2 + c_3\lambda_2^2)(1 - e^{-\lambda/2}(1 - \lambda_2 - r\lambda'_2))) \quad (\text{B.13})$$

$$1 - e^{-\lambda} - re^{-\lambda}\nu' = -m^2r^2(2\lambda_2 + c_2\lambda_2^2 + (1 + 2c_2\lambda_2 + c_3\lambda_2^2)(1 - e^{-\nu/2})) \quad (\text{B.14})$$

$$r\nu'(1 + 2c_2\lambda_2 + c_3\lambda_2^2) = 4(e^{\lambda/2} - 1)(1 + c_2\lambda_2 + (c_2 + c_3\lambda_2)(1 - e^{-\nu/2})). \quad (\text{B.15})$$

After linearizing in ν , λ , μ , equations (B.13, B.14, B.15) can be solved exactly:

$$\begin{aligned} \nu &= -ce^{-mr}\frac{1}{r}, \quad \lambda = \frac{c}{2}e^{-mr}\left(\frac{1}{r} + m\right), \\ \mu &= \frac{c}{2m^2}e^{-mr}\left(\frac{1}{r^3} + \frac{m}{r^2} + \frac{m^2}{r}\right), \end{aligned} \quad (\text{B.16})$$

manifesting the vDVZ discontinuity. When $mr_g \ll 1$, we have $c = 4r_g/3$ where $r_g = M/4\pi$.

Without linearizing, equations (B.13, B.14, B.15) cannot be solved as written. Although we do have three equations for the three unknowns, it is seen that the only derivatives of the unknowns in these equations are λ' and λ'_2 in eq.(B.13), ν' in eq.(B.14), and ν' in eq.(B.15). What makes it possible to solve the system is to derive an algebraic relation between r , ν , λ , λ_2 by equating the two expressions for ν' . Then one can select any two of the three unknowns ν , λ , λ_2 and derive a system of two first-order differential equations for the two selected unknowns.

However, even in the current form one can use Vainshtein approximation to extract a solution which is valid well inside the Vainshtein radius $r \ll r_V = (r_g/m^2)^{1/3}$, but before getting too close to the horizon, where λ_0 diverges as $(1 - r_g/r)^{-1/2}$ and $T_\nu^\mu \sim m^2\lambda_0$ can no longer be neglected from the Einstein equations.

When $r_g \ll r \ll r_V$, we substitute the linearized Schwarzschild solution $\nu = -\lambda = -r_g/r$ into (B.15), also linearized in ν and λ , but exact in λ_2 , and solve for λ_2 . We find

$$\lambda_2^2 = 1/c_3. \quad (\text{B.17})$$

which implies $c_3 > 0$. Numerical integration shows that the positive root is connected to the asymptotically decaying solution.

When $r \rightarrow r_g$ but still $Gm^2(1 - r_g/r)^{-1/2} \ll 1$, and assuming that λ_1 and λ_2 remain finite we expect the geometry to be close to Schwarzschild. Substituting $e^\nu = e^{-\lambda} = 1 - r_g/r$ in (B.15) and solving for $\lambda_2(r)$ gives, at $r = r_g$ ¹²:

$$\lambda_2 = -2 - c_2/c_3 + \sqrt{4 + c_2^2/c_3^2 - 1/c_3}, \quad (\text{B.18})$$

$$r_g(1 - r_g/r)^{1/2}\lambda_2' = -\lambda_2 - 2c_2/\sqrt{4c_3^2 + c_2^2 - c_3}. \quad (\text{B.19})$$

The same substitution $e^{-\lambda} = 1 - r_g/r$ in (B.11) gives, at $r = r_g$,

$$\lambda_1 = 1 + r_g(1 - r_g/r)^{1/2}\lambda_2'. \quad (\text{B.20})$$

These a posteriori justify our assumption of $\lambda_{1,2}$ remaining finite. In the limit $m \rightarrow 0$ these approximate solutions are expected to become exact, which is confirmed by numerical integration.

B.1 Stress-energy components

The energy density and stresses in various regions of interest can be obtained from the above approximate solutions:

When $r_V \ll r \ll m^{-1}$, it is seen from eqs. (B.11), (B.12), and (B.16) that $\lambda_2 \simeq \mu/2$ and $\lambda_1 \simeq \lambda_2 + r\lambda_2'$ are the largest eigenvalues, so we can approximate

$$T_0^0 = m^2\mathcal{O}(r_g/r), \quad (\text{B.21})$$

$$T_1^1 \simeq -2T_2^2 \simeq -\frac{2}{3}m^2\left(\frac{r_V}{r}\right)^3. \quad (\text{B.22})$$

When $r_g \ll r \ll r_V$, still $\lambda_1 \simeq \lambda_2 \simeq 1/\sqrt{c_3}$ are the largest eigenvalues, and we have

$$T_0^0 \simeq -m^2(4 + 3c_2/\sqrt{c_3})/\sqrt{c_3}, \quad (\text{B.23})$$

$$T_1^1 \simeq T_2^2 \simeq -m^2(2 + c_2/\sqrt{c_3})/\sqrt{c_3}. \quad (\text{B.24})$$

Finally, when $r \rightarrow r_g$ and $\lambda_0 \rightarrow -\infty$

$$T_0^0 = m^2\mathcal{O}(1), \quad (\text{B.25})$$

$$T_1^1 \simeq -m^2\lambda_0(1 + 2c_2\lambda_2 + c_3\lambda_2^2), \quad (\text{B.26})$$

$$T_2^2 \simeq -m^2\lambda_0(1 + c_2(\lambda_1 + \lambda_2) + c_3\lambda_1\lambda_2), \quad (\text{B.27})$$

where $\lambda_{1,2}$ are given in (B.18) and (B.20). Pressure $p = -(T_1^1 + 2T_2^2)/3$ diverges at horizon unless $c_2^2 = c_3$. Note that since the energy density T_0^0 is negative on average, $p \rightarrow -\infty$ at horizon should not be very surprising – reversing the sign of the action does not change the equations of motion, but it reverses the sign of stress-energy tensor.

¹²The larger root was chosen for λ_2 because in the “Vainshtein region” $\lambda_2 = 1/\sqrt{c_3}$ which is larger than both roots in the parameter range of interest $c_2^2 \leq c_3$.

C Near-hole solutions in FP2

As argued in §3.3.2 the quasi-stationary near-hole solutions can be obtained from $\nabla_\mu T^\mu_\nu = 0$, or equivalently, from the equations of motion for ϕ^a on Schwarzschild geometry. To derive these equations, we first need to determine the eigenvalues λ_a of the matrix $\delta_b^a - \sqrt{H_b^a}$, in terms of which the FP action is defined. For the spherically symmetric metric

$$ds^2 = CdT^2 - 2DdTdR - AdR^2 - BR^2d\Omega^2. \quad (\text{C.1})$$

they are given by

$$\lambda_\pm = 1 - \frac{1}{\sqrt{2\Delta}} \left[A + C \pm \sqrt{(A - C)^2 - 4D^2} \right]^{1/2}, \quad (\text{C.2})$$

$$\lambda_2 = \lambda_3 = 1 - B^{-1/2}, \quad (\text{C.3})$$

where

$$\Delta = AC + D^2, \quad (\text{C.4})$$

and we have renamed $\lambda_0 = \lambda_+$ and $\lambda_1 = \lambda_-$.¹³ Since the FP2 action (23) depends on λ_\pm through the combinations $\lambda_+ + \lambda_-$ and $\lambda_+\lambda_-$, it is useful to define

$$F = \frac{1}{\sqrt{2}} \left(\left[A + C + \sqrt{(A - C)^2 - 4D^2} \right]^{1/2} + \left[A + C - \sqrt{(A - C)^2 - 4D^2} \right]^{1/2} \right), \quad (\text{C.5})$$

in terms of which

$$\lambda_+ + \lambda_- = 2 - F/\sqrt{\Delta}, \quad (\text{C.6})$$

$$\lambda_+\lambda_- = 1 + (1 - F)/\sqrt{\Delta}. \quad (\text{C.7})$$

The FP2 Lagrangian becomes

$$U = -\frac{F}{\sqrt{\Delta}}\beta + \frac{1}{\sqrt{\Delta}}\tilde{\beta} + \beta + 2\lambda_2 + c_2\lambda_2^2, \quad (\text{C.8})$$

where we defined the new parameters

$$c_2 = 1 + \tilde{c}_2, \quad c_3 = \tilde{c}_2 + \tilde{c}_3, \quad (\text{C.9})$$

and the functions

$$\beta \equiv 1 + 2c_2\lambda_2 + c_3\lambda_2^2, \quad \tilde{\beta} \equiv 1 + 2\tilde{c}_2\lambda_2 + \tilde{c}_3\lambda_2^2. \quad (\text{C.10})$$

The stress-energy tensor components can now be calculated by varying the action with respect to the metric:

$$m^{-2}T^\nu_\mu = -2g_{\mu\sigma} \frac{\partial U}{\partial g_{\nu\sigma}} - \delta^\nu_\mu U. \quad (\text{C.11})$$

¹³One can see from these expressions why $T^{01} \propto \delta S_{FP}/\delta D$ (and similarly T_{01}) is proportional to D with a regular proportionality coefficient at $D = 0$. S_{FP} depends on D via λ_\pm , and $\sqrt{\Delta}$ in the overall $\sqrt{-g}$. Both are quadratic in D except when: (i) $A = C$, in which case since $\lambda_+ = \lambda_-$ at $D = 0$ the symmetry of S_{FP} in λ_\pm ensures that terms linear in D cancel. (ii) $AC = 0$ which is a singular point at $D = 0$ and therefore cannot be approached from $D = 0$ side, which is dictated by the asymptotic condition.

For instance, we get

$$m^{-2}T_0^1 = C \frac{\partial U}{\partial D} - 2D \frac{\partial U}{\partial A} = \frac{\beta D}{\sqrt{\Delta} F}. \quad (\text{C.12})$$

After deriving the stress-energy components one can use the Vainshtein approximation (53), namely,

$$C = 1 - r_g/r, \quad A = C^{-1}(\gamma^{-2} - D^2), \quad (\text{C.13})$$

to express everything in terms of r and the two unknowns λ_2 and D . In particular, using $R = B^{-1/2}r$, we get

$$\frac{1}{\sqrt{\Delta}} = \gamma = dR/dr = 1 - \lambda_2 - r\lambda_2'. \quad (\text{C.14})$$

The equations for D and λ_2 can be derived from the stress-energy conservation

$$\sqrt{-g}\nabla_\mu T_\nu^\mu = \partial_\mu(\sqrt{-g}T_\nu^\mu) - \frac{1}{2}\sqrt{-g}T^{\alpha\beta}\partial_\nu g_{\alpha\beta} = 0, \quad (\text{C.15})$$

which is particularly simple for the $\nu = 0$ component in the quasi-stationary approximation. It becomes the condition of steady energy flux, $\partial_1(\sqrt{-g}T_0^1) = 0$, whose integration constant is (minus) the accretion rate $\dot{r}_g = -\alpha m^2 r_g^2$. Using the expression (C.12), we obtain

$$\frac{D\beta}{F} = \frac{\alpha r_g^2}{BR^2} = \frac{\alpha r_g^2}{r^2}. \quad (\text{C.16})$$

Another independent equation can be obtained from the $\nu = 1$ component of (C.15) which together with the $\nu = 0$ component form a linear system of equations for D' and λ_2'' . However, it is technically easier to derive this second equation by using the Vainshtein approximation directly inside the FP2 Lagrangian (C.8) and writing it as an action for the scalar fields ϕ^a on Schwarzschild background:

$$S = \int dr r^2 U. \quad (\text{C.17})$$

Here U is a function only of r , D , λ_2 , and γ . As a consistency check, note that since $\phi^{0'} = D\gamma/C$ the equation of motion for ϕ^0 is given by

$$\frac{\delta S}{\delta \phi^0} = -\partial_r \left(r^2 \frac{C}{\gamma} \frac{\partial U}{\partial D} \right) = -\partial_r \left(\frac{D\beta r^2}{F} \right) = 0, \quad (\text{C.18})$$

which integrates to our first equation (C.16). The equation of motion for $\phi^i = (1 - \lambda_2)rn^i$ can be obtained by varying S with respect to λ_2 , but remembering that ϕ^0 also depends on λ_2 via $\gamma = 1 - (r\lambda_2)'$. Subtracting that contribution, we get

$$r^2 \frac{\partial U}{\partial \lambda_2} + r \partial_r \left[r^2 \left(\frac{\partial U}{\partial \gamma} - \frac{D}{\gamma} \frac{\partial U}{\partial D} \right) \right] = 0, \quad (\text{C.19})$$

which after some algebra, and using the identity

$$CF^2 + D^2 = \gamma^{-2}(1 + \gamma C)^2, \quad (\text{C.20})$$

yields

$$r\partial_r \left[\frac{\beta r^2(1 + \gamma C)}{\gamma F} - \tilde{\beta} r^2 \right] = r^2(1 - \gamma F) \frac{\partial \beta}{\partial \lambda_2} + r^2 \gamma \frac{\partial \tilde{\beta}}{\partial \lambda_2} + 2r^2(1 + c_2 \lambda_2). \quad (\text{C.21})$$

This equation contains both λ_2'' and D' , but it becomes algebraic in λ_2 once we eliminate D using

$$F = (\gamma^{-1} + C) \left(C + \frac{\alpha^2 r_g^4}{\beta^2 r^4} \right)^{-1/2}, \quad (\text{C.22})$$

which can be obtained by substituting $D = \alpha r_g^2 F / r^2 \beta$ from (C.16) into the identity (C.20). After this substitution, the left hand side of (C.21) which would have contained λ_2'' and D' becomes

$$r\partial_r \left[(C\beta^2 r^4 + \alpha^2 r_g^4)^{1/2} - \tilde{\beta} r^2 \right]. \quad (\text{C.23})$$

It is manifestly independent of λ_2'' . Factors of λ_2' will also cancel from the two sides, and we finally get

$$\beta \left(1 - \frac{3r_g}{4r} \right) + (c_2 + c_3 \lambda_2)(1 + (1 - \lambda_2)C) = \left(C + \frac{\alpha^2 r_g^4}{\beta^2 r^4} \right)^{1/2} (1 + c_2 \lambda_2 + 2(c_2 + c_3 \lambda_2)). \quad (\text{C.24})$$

For given α and r_g , λ_2 is solved for from (C.24), using which D is given by

$$D = \frac{\alpha r_g^2}{\beta r^2} (\gamma^{-1} + C) \left(C + \frac{\alpha^2 r_g^4}{\beta^2 r^4} \right)^{-1/2} \quad (\text{C.25})$$

which is obtained by substituting (C.22) in (C.16).

D Transition region

It is difficult to derive analytic results for the transition region $r \sim r_V = (r_g/m^2)^{1/3}$, but since accretion effects rapidly disappear away from the horizon one can safely linearize in D and use the knowledge about static solution in this region. After this linearization, the only component of the stress tensor that depends on D is (c.f. eq.(C.12))

$$T_0^1 \simeq m^2 \frac{D\beta}{\sqrt{\Delta}(\sqrt{A} + \sqrt{C})}, \quad (\text{D.1})$$

the rest being given by their $\alpha = 0$ expressions. Here $\Delta \simeq AC$ and the Vainshtein approximation (C.13) is not used, so (D.1) is valid even for $r > r_V$.

Inside the Vainshtein radius $r \ll r_V$, we have, from eq.(C.16), a steady flow with energy flux

$$\sqrt{\Delta} T_0^1 = \alpha m^2 \frac{r_g^2}{r^2}, \quad (\text{D.2})$$

corresponding to $\dot{r}_g = -\alpha m^2 r_g^2$, which in turn fixes the linearized solution (43) of §3.3.1 and from (D.1) we get for $r_V \ll r \ll m^{-1}$

$$\sqrt{\Delta} T_0^1 \simeq m^2 \frac{d}{2} = \frac{2}{3} \alpha m^2 \frac{r_g^2}{r^2}. \quad (\text{D.3})$$

So there is a net positive flux of energy into the transition region. This can be understood in the following way. On a static solution, the relevant degree of freedom in this region is $R/r = B^{-1/2}$, or the related quantity λ_2 (see eqs.(B.12), (B.16), (B.17))

$$\lambda_2 \simeq \begin{cases} \frac{1}{3}(r_V/r)^3 & r_V \ll r \ll m^{-1}, \\ 1/\sqrt{c_3} & r_g \ll r \ll r_V, \end{cases} \quad (\text{D.4})$$

which is approximately a function only of the ratio r/r_V . The energy density profile is therefore a function of r/r_V ; as was shown in appendix B.1, in going from $r \gg r_V$ to $r \ll r_V$ it interpolates between a negligible amount to a negative constant $\varepsilon \sim -m^2$. Once we allow accretion, r_g gradually decreases and as a result the Vainshtein radius and the above-mentioned profile adiabatically shrink to smaller radii. Therefore, the total amount of energy in this region $\Delta M \sim \varepsilon r_V^3 \sim -r_g$ grows, requiring a net positive energy flux.

The exact value of ΔM can be found from the difference of the invariant masses M_1 and M_2 inside two spheres of radii r_1 and r_2 , where $r_g \ll r_1 \ll r_V$ and $r_V \ll r_2 \ll m^{-1}$. Since at distances $r \gg r_g$ the geometry is nearly flat, the metric can be transformed into $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} \ll 1$, in terms of which the invariant mass inside a sphere of radius r is given by

$$M = \frac{1}{2} \int (\partial_i h_{jj} - \partial_j h_{ij}) n^i r^2 d\Omega. \quad (\text{D.5})$$

Using circumference defined radius to parametrize the static metric [as in (B.2)], we get $h_{ij} = -\lambda n^i n^j$, where inside the Vainshtein radius λ is given by the Einsteinian r_g/r expression, and outside by the vDVZ one $2r_g/3r$. The enclosed masses are then

$$M_1 = 4\pi r_g, \quad M_2 = \frac{8\pi}{3} r_g, \quad (\text{D.6})$$

giving

$$\Delta M = M_2 - M_1 = -\frac{4\pi}{3} r_g. \quad (\text{D.7})$$

Therefore, when $\alpha \neq 0$ the time-dependence of ΔM exactly accounts for the difference between (D.2) and (D.3).

Similar methods can be used to find D throughout the transition region. One uses the numerical static solution of [17] to determine $M(r, r_g)$. By numerically differentiating $M(r, r_g)$ with respect to r_g , one can find

$$\dot{M}(r, r_g) = \dot{r}_g \partial_{r_g} M(r, r_g). \quad (\text{D.8})$$

The energy flux at any $r_2 \sim r_V$ is determined from

$$\dot{M}(r_2) - \dot{M}_1 = -r^2 \sqrt{\Delta} T_0^1 \Big|_{r_1}^{r_2}, \quad (\text{D.9})$$

using which one solves for D from (D.1).

E The mass of a star and the global charge Q_0

Consider a star of gravitational radius r_g deep inside the Vainshtein regime. In the Schwarzschild coordinates, where the metric is given by

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\Omega^2, \quad (\text{E.1})$$

we have $e^\nu = e^{-\lambda} = 1 - r_g/r$ for $r \ll r_V$. We will show in what follows that for finite redshift stars the Noether charge Q_0 coincides with minus the Schwarzschild mass $4\pi r_g$ of the star.

Note first that for all $r \gg r_g$ the functions ν and λ are infinitesimal (although different from the Einsteinian value when $r \gg r_V$), so that the space-time is nearly flat. Thus, the ADM mass within any sphere of radius $r \gg r_g$ is well-defined. Consider a sphere of radius r_1 where $r_g \ll r_1 \ll r_V$. The enclosed ADM mass coincides with the Schwarzschild mass $4\pi r_g$. On the other hand the ADM mass inside a sphere of much larger radius, $r \gg m^{-1}$, must vanish in massive gravity; therefore, the contribution of S_{FP} to M_{ADM} coming from the region $r_1 < r < \infty$ should be exactly $-4\pi r_g$. Given the closeness of the metric (E.1) to Minkowski in this region, this contribution of S_{FP} can be directly calculated from the volume integral of its energy density, namely

$$\int_{r_1 \gg r_g}^{\infty} d^3\mathbf{r} T_t^t = -4\pi r_g. \quad (\text{E.2})$$

We will show that the integral on the left coincides with the volume integral of J_0^t in the same region of space, which is what one expects if Q_0 is a good characterization of the energy of S_{FP} :

Since for $r \gg r_g$ the unitary-gauge metric is approximately diagonal (even if there is time-dependence), we have $\phi^0 \simeq t$ and $\partial_t \phi^i \simeq 0$, so J_0^t and T_t^t can be replaced with the unitary-gauge expressions J_0^0 and T_0^0 . From (76) we have (after redefining $m^2 U \rightarrow U$)

$$J_0^0 = T_0^0 - U, \quad (\text{E.3})$$

and the above requirement on Q_0 implies

$$\int_{r_1 \gg r_g}^{\infty} dr r^2 U \ll r_g. \quad (\text{E.4})$$

This is trivially satisfied in the linear regime $r \gg r_V$, where the perturbations of the unitary-gauge metric h_{ab} are themselves infinitesimal, and $U = \mathcal{O}(m^2 h_{ab}^2)$.

Now consider the contribution from a region $r_1 < r < r_2$ where $r_g \ll r_1 \ll r_V \ll r_2 \ll m^{-1}$. The static solution is a good approximation throughout this region and we can use the formalism of appendix B to verify eq.(E.4). Since $\nu \ll 1$, the stress-energy conservation (B.6) simplifies to

$$T_1^{1'} \simeq \frac{2}{r}(T_2^2 - T_1^1), \quad (\text{E.5})$$

and for a diagonal unitary-gauge metric, T_1^1 and T_2^2 of a generic FP (including FP2) are given by

$$T_1^1 = (\lambda_1 - 1) \frac{\partial U}{\partial \lambda_1} - U, \quad T_2^2 = \frac{1}{2}(\lambda_2 - 1) \frac{\partial U}{\partial \lambda_2} - U, \quad (\text{E.6})$$

(the factor of $1/2$ in T_2^2 is because $\lambda_2 = \lambda_3$). Moreover, in this region $\lambda \ll \lambda_2$, and the expression (B.11) for λ_1 can be approximated as

$$\lambda_1 \simeq (\lambda_2 r)', \quad (\text{E.7})$$

while $\lambda_0 \simeq \nu/2$ and its derivatives can be neglected. Substituting (E.6) and (E.7) in (E.5) yields

$$r \frac{\partial U}{\partial \lambda_2} = \left(r^2 \frac{\partial U}{\partial \lambda_1} \right)'. \quad (\text{E.8})$$

We can now transform

$$\begin{aligned} \int_{r_1}^{r_2} dr r^2 U &= \frac{1}{3} r^3 U \Big|_{r_1}^{r_2} - \frac{1}{3} \int_{r_1}^{r_2} dr r^3 U' \\ &= \frac{1}{3} r^3 U \Big|_{r_1}^{r_2} - \frac{1}{3} \int_{r_1}^{r_2} dr r^3 \left(\frac{\partial U}{\partial \lambda_2} \lambda_2' + \frac{\partial U}{\partial \lambda_1} (\lambda_2 r)'' \right), \end{aligned} \quad (\text{E.9})$$

and use (E.8) to obtain

$$\int_{r_1}^{r_2} dr r^2 U = \frac{1}{3} r^3 \left(U - r \lambda_2' \frac{\partial U}{\partial \lambda_1} \right) \Big|_{r_1}^{r_2}. \quad (\text{E.10})$$

But r_2 is in the linear regime where $\lambda_a \sim h_{aa} \ll 1$, and hence the right hand side which is quadratic in λ_a is negligible at the upper limit. At the lower limit, since $U \propto m^2$, $\lambda_{1,2} \sim 1$, and $r_1^3 \ll r_V^3 = r_g/m^2$, we get the desired relation (E.4).

For a star of finite redshift all eigenvalues λ_a remain finite and the contribution of the region $r < r_1$ to Q_0 , which is of order $r_1^3 J_0^0 \sim m^2 r_1^3$, is much less than r_g . Therefore, Q_0 is given by the total amount of negative mass in the region $r \gg r_g$ evaluated by the volume integral of T_0^0 and we have $Q_0 = -4\pi r_g$.

F Fluid accretion

Potential flow of ideal fluids can be described by a single scalar field ϕ with the k-essence action [34]

$$S = \int d^4x \sqrt{-g} P(X), \quad (\text{F.1})$$

where $X = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. The stress-energy tensor is

$$\mathcal{T}_{\mu\nu} = 2P_X \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} P, \quad (\text{F.2})$$

where $P_X = \partial P / \partial X$. We can therefore identify the fluid pressure p , energy density ε , and velocity u_μ as

$$p = P, \quad \varepsilon = 2P_X X - P, \quad u_\mu = \partial_\mu \phi / \sqrt{X}. \quad (\text{F.3})$$

The $p = c_s^2 \varepsilon$ equation of state, therefore, corresponds to

$$P = X^n, \quad \text{with} \quad n = \frac{1 + c_s^{-2}}{2}. \quad (\text{F.4})$$

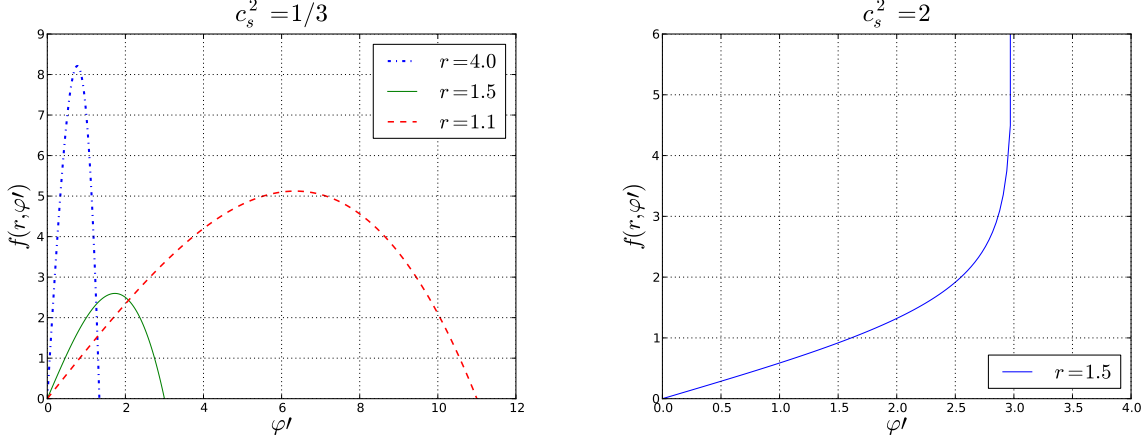


Figure 2: $f(r, \varphi')$ plotted as a function of φ' at fixed values of r and for $r_g = 1$. When $c_s^2 < 1$ (left), the maximum of $f(r, \varphi')$ decreases until a critical radius is reached and then increases. When $c_s^2 > 1$ (right), $f(r, \varphi')$ has no maximum.

For static spherically symmetric fluids with asymptotic density ε_0 , we have $\phi = \phi_0 t$, with $\phi_0 = (c_s^2 \varepsilon_0)^{1/(1+c_s^{-2})}$, which on Schwarzschild geometry gives the singular at horizon solution (61). To have an accreting solution we take $\phi = \phi_0(t + \varphi(r))$. Integrating once the ϕ equation of motion, we get

$$f(r, \varphi') \equiv r^2 \left(1 - \frac{r_g}{r}\right) \varphi' \left[\left(1 - \frac{r_g}{r}\right)^{-1} - \left(1 - \frac{r_g}{r}\right) \varphi'^2 \right]^{n-1} = A, \quad (\text{F.5})$$

where A is an integration constant. For any A , there exists at large values of r a decaying solution $\varphi' \approx A/r^2$, corresponding to the steady accretion

$$\mathcal{T}_t^r = -(1 + c_s^2) \varepsilon_0 \frac{A}{r^2}. \quad (\text{F.6})$$

However, when $c_s^2 \leq 1$ only one of the decaying solutions matches a regular solution at horizon (the Bondi solution). This can be shown as follows. At any fixed r the function $f(r, \varphi')$ in (F.5) is maximized at

$$\varphi' = c_s \left(1 - \frac{r_g}{r}\right)^{-1}, \quad (\text{F.7})$$

giving a maximum $f_{\max}(r)$. As r is decreased from infinity, $f_{\max}(r)$ reaches a minimum at the critical radius

$$r_c = \frac{3 + c_s^{-2}}{4} r_g, \quad (\text{F.8})$$

and then increases (figure 2). For $A > A_{\text{Bondi}} = f_{\max}(r_c)$, the asymptotically decaying solution ceases to exist below some radius $r(> r_c)$. For $A < A_{\text{Bondi}}$, there exist a solution all the way to the horizon but φ' is not large enough there. Therefore, X diverges as $(1 - r_g/r)^{-1}$, leading to a pressure singularity. Only for $A = A_{\text{Bondi}}$ the asymptotically decaying solution connects to a rapidly falling solution at horizon, where $\varphi' \rightarrow (1 - r_g/r)^{-1}$ and X remains finite.

When $c_s^2 > 1$, $f(r, \varphi')$ is unbounded above and there is no critical point as can be seen from figure 2. For all A the asymptotically decaying solution goes into

$$\varphi' = \left(1 - \frac{r_g}{r}\right)^{-1} - \frac{1}{2} \left(\frac{A}{r_g^2}\right)^{-2/(1-c_s^{-2})}, \quad (\text{F.9})$$

in the $r \rightarrow r_g$ limit, and has a finite pressure. The energy of fluid elements at horizon reaches the constant

$$e = u_0 = \left(\frac{A}{r_g^2}\right)^{1/(1-c_s^{-2})}. \quad (\text{F.10})$$

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